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ON THE PLANAR MOTION MECHANISM USED IN SHIP MODEL TESTING

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CONTENTS

In the linear theory of small departures from steady reference motions of submarines and ships it is standard practice to employ the idea of hydrodynamic 'derivatives'. These derivatives permit the magnitudes of fluid forces and moments to be specified. In recent years it has become common to measure the derivatives by means of a 'planar motion mechanism' which is essentially a device for oscillating a ship (or submarine) model while it is being towed in a testing tank.

The derivatives referred to in the maritime literature have invariably been 'slow motion derivatives'. The theory of the planar motion technique is recast in terms of 'oscillatory derivatives'—or, better, 'oscillatory coefficients', since they are more appropriate for use where the mechanism is concerned. The idea behind these quantities is borrowed from aeronautical practice, but it requires some adaptation because (*a*) ship models work at the water surface, and (*b*) ships and submarines are subject to significant buoyancy forces. There can be little doubt that the planar motion mechanism is a powerful tool and a reappraisal is perhaps timely since the first mechanism of this sort to be installed in the U.K. has recently been commissioned (1968).

Introduction

In recent years it has become common to test models of ships and submarines using a 'planar motion mechanism'. This is a device that imparts a pure sinusoidal motion in one degree of freedom—in yaw for instance—to a model that is towed along a testing tank. While this motion is executed, measurements are made of the forces acting on the model, whence the fluid forces may be deduced. The technique is of fairly recent origin, having been pioneered in the U.S.A. by Gertler (1959) and Goodman (1960), but there is already no doubt as to its value in the measurement of hydrodynamic forces and, in particular, of the hydrodynamic 'derivatives' with which this paper is largely concerned. It is at the same time more versatile and more economical in use than alternative techniques.

The value of the planar motion mechanism rests on its assumed ability to impose sinusoidal motions that are pure and inexorable. In reality, of course, this is not strictly possible and it is conceivable that, for extreme accuracy, an alternative approach might be needed. In theory at least, it may be better from the point of view of accuracy to measure the impressed forces *and the motions* so that the analysis has to be based on the coupled equations of motion (rather than on the equations taken one at a time). But the case for such a sophisticated approach has by no means been made where ship models are concerned and the planar motion mechanism is probably the most promising practical proposition at the present time.

In this paper the theoretical background of the planar motion mechanism is presented in a new way. The concept of the 'oscillatory derivative' is adapted for this purpose (although for reasons that will be explained the name 'oscillatory coefficient' is preferable and will be adopted here.) For while oscillatory derivatives (or coefficients) are familiar in aeronautical practice, they appear not to have found any place whatsoever in the maritime literature. This fresh approach is thought to have intrinsic merit and it also suggests a line of speculation that may be of some significance. Thus if oscillatory coefficients could be found for a sufficient range of frequency they could be used in conjunction with Fourier integral techniques for the study of transient behaviour of ships and submarines, even though in most cases the response of ships and submarines is much too slow to warrant the use of such a technique. Moreover an understanding of the frequency dependence of the oscillatory coefficients can be valuable in interpreting experimental results which are obtained by tests conducted at several different frequencies.

Notation

C	centre of mass of model
F, P	forces applied to model
Fn	Froude number
G, H	moments applied to model
h	metacentric height of submarine
I_y, I_z	moments of inertia of model about pitch and yaw axes respectively
$\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$	unit vectors in directions of body axes Cx, Cy, Cz; Cxz is a plane of symmetry, and Cxy is parallel to the undisturbed water surface

L, M, N	moments of fluid force about C parallel to $\hat{\imath}, \hat{\jmath}, \hat{k}$
l	length of model 'between perpendiculars'
l_0	see figures 3, 5
m	mass of model
p, q, r	perturbations of components of angular velocity in directions $\hat{\imath}, \hat{\jmath}, \hat{k}$; i.e. angular velocities of roll, pitch and yaw
Re	Reynolds number
T	axial torque applied to model
t	time
$\boldsymbol{U}$	velocity of centre of mass
$\overline{U}$	reference velocity in direction $\hat{\imath}$
u, v, w	perturbations of components of velocity of C in directions $\hat{\imath}, \hat{\jmath}, \hat{k}$; i.e. velocities of surge, drift (or sway) and heave
X, Y, Z	components of fluid force in the $\hat{\imath}, \hat{\jmath}, \hat{k}$ directions
$\bar{x}, \bar{z}$	coordinates of mass centre of model in appendixes 2 and 3
ϵ	phase difference (see equation (41))
ζ, η	angles of rudder and hydroplane deflection
ν	dimensionless (or 'reduced') frequency $= \omega l/\overline{U}$ (a second dimensionless frequency is used in figures 8 and 9, namely $\omega' = \omega\sqrt{(l/g)}$)
θ, ϕ, ψ	small angles of pitch, roll and yaw
$\boldsymbol{\Omega}$	angular velocity of ship or submarine
ω	frequency of oscillation (rad s^{-1})

Subscripts and dressings

*	steady state value
0	amplitude (except l_0)
1, 2	forward and after
in	in phase with displacement or orientation
out	in quadrature with displacement or orientation
~	oscillatory
′	dimensionless

1. 'Slow motion derivatives' and 'oscillatory coefficients'

Departure from steady motion in a straight line

A number of writers have discussed problems of directional stability and control of ships and submarines in calm water (see, for example, Abkowitz 1964). The problems that they seek to elucidate arise from the fact that—to take a surface ship as an example—small departures from a steady reference motion

$$\boldsymbol{U} = \overline{U}\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \tag{1}$$

are associated with small variations of the hydrodynamic forces and moments. As is particularly well known to aeronautical engineers (who are faced with comparable problems) this means that the stability of the reference motion is open to question. To take one component of the fluid force by way of illustration consider the component $Y\hat{j}$. A small departure from the motion (1) will produce a variation ΔY in Y.

In general the value of ΔY at any instant t depends on the parameters defining the instantaneous motion together with the history of the departure from the reference motion. That is to say, ΔY is a function of the instantaneous components of velocity, acceleration and, where appropriate, displacement of the ship, together with their previous values. Thus if the motion at any instant is

$$\left.\begin{aligned} \boldsymbol{U} &= (\bar{U}+u)\,\hat{\imath}+v\hat{\jmath}+w\hat{k}, \\ \boldsymbol{\Omega} &= p\hat{\imath}+p\hat{\jmath}+r\hat{k}, \end{aligned}\right\} \tag{2}$$

and if, in the interests of brevity, the symmetric variables u, w, q are omitted from the discussion of the antisymmetric quantity† ΔY, then the increment of force ΔY will be of the form

$$\Delta Y = f(\phi, v, p, r, \dot{v}, \dot{p}, \dot{r}, t;\ \text{previous values of } \phi, v, \ldots, \dot{r};\ \bar{U} \text{ and other constant parameters}), \tag{3}$$

where the symbols have the meanings given in the list of notation.

Fortunately it is possible to simplify the form of the function (3) for most of the small disturbed motions with which one is normally concerned. Consider, for example, the problem of specifying v at any instant $t-\tau$ during the disturbed motion (i.e. $\tau < t-t_0$), where t_0 denotes the time of onset of the disturbed motion). It is shown in Appendix 1 that for disturbances of the form $v = v_0 e^{\mu t} \sin(\omega t+\epsilon)$, which are typical of the motions with which we are normally concerned, the value of v at the instant $t-\tau$ can be represented by the infinite Taylor series,

$$v(t-\tau) = v(t) - \tau\dot{v}(t) + \frac{\tau^2}{2!}\ddot{v}(t) - \frac{\tau^3}{3!}\dddot{v}(t) + \ldots. \tag{4}$$

That is, the disturbed velocity at some instant $t-\tau$ is determined by the values of $v, \dot{v}, \ddot{v}, \ldots$, at the time t.

Similar expressions can be formed for the values of all the parameters $\phi, v, \ldots, \dot{r}$ at any time $t-\tau$ in the interval $t_0 < t-\tau < t$. The previous history of the given type of disturbed motion can therefore be fixed by specifying the instantaneous values of $\phi, v, \ldots, \dot{r}$ and all their higher derivatives with respect to time. In these circumstances relation (3) can be rewritten in a simpler form as

$$\Delta Y = f(\phi, v, p, r, \dot{v}, \dot{p}, \dot{r}, \ddot{v}, \ddot{p}, \ddot{r}, \dddot{v}, \ldots, t), \tag{5}$$

where the form of the function $f(\ )$ depends in part on the value of $\bar{U}$ and other appropriate constant parameters.

Expression (5) for ΔY is similar to the one normally used in formulating problems of ship dynamics, except that instead of neglecting the history of the motion we have allowed for it, at least for exponentially growing or decreasing oscillatory motion, by including the higher order derivatives of v, p and r. We have therefore a relation for ΔY as a function of all the variables

$$\phi, v, \ldots, \dot{r}, \ddot{v}, \ldots, \dddot{v}, \ldots, t.$$

In most practical cases the conventional approach of expanding ΔY in terms of a Taylor series can be adopted. Thus for small disturbances the component ΔY is specified 'to the first order' by

$$\begin{aligned} \Delta Y = {} & Y_\phi \phi + Y_p p + Y_{\dot{p}} \dot{p} + Y_{\ddot{p}} \ddot{p} + \ldots \\ & + Y_v v + Y_{\dot{v}} \dot{v} + Y_{\ddot{v}} \ddot{v} + \ldots \\ & + Y_r r + Y_{\dot{r}} \dot{r} + Y_{\ddot{r}} \ddot{r} + \ldots \\ & + Y(t), \end{aligned} \tag{6}$$

† In a linear theory the independence of the symmetric and antisymmetric motions is readily justified.

where the Taylor series has been curtailed in the usual way to exclude all nonlinear terms and where $Y(t)$ represents any time-dependent forcing. In this equation the coefficients are defined by expressions such as

$$Y_v = \left|\frac{\partial \Delta Y}{\partial v}\right|_{\text{steady state}}, \quad Y_{\dot{v}} = \left|\frac{\partial \Delta Y}{\partial \dot{v}}\right|_{\text{steady state}}. \tag{7}$$

Here 'steady state' means the condition in which $\boldsymbol{U} = \bar{U}\boldsymbol{\hat{\imath}}$, $\boldsymbol{\Omega} = 0$ and the partial differentials indicate, for example, that Y_v is associated with an increment ΔY generated by a perturbation v alone, with the remaining motion parameters held constant at their steady-state zero values.

Conventionally an approximate form of equation (6) is found for ΔY. This rests upon the idea of 'derivatives', which may be of two types—namely 'slow motion derivatives' and 'oscillatory coefficients'. Both types will now be discussed briefly. It will be understood that ΔY is here used only by way of example; similar arguments apply to ΔX, ΔZ, ΔL, ΔM and ΔN.

Slow motion derivatives

The concept of 'slow motion' has been discussed in some detail in an aerodynamic context by Duncan (1952). Consider, by way of example, a sway displacement y of a ship, such that $v = \dot{y}, \dot{v} = \ddot{y}, \ldots$. The sway motion of the ship can be regarded as 'slow', if it does not vary rapidly with time. We can express this property by postulating that, at any instant t, v/y, $\dot{v}/v$, $\ddot{v}/\dot{v}$ are all very much less than $\bar{U}/l$, where l is the length of the ship (or any other convenient parameter with dimensions of length). Thus not only are the parameters $y, v, \dot{v}, \ldots$ small, but their relative rates of change are small. For example, v, the instantaneous rate of change of y, and y are associated with a time scale which is very much larger than $l/\bar{U}$, the time for the ship to move through its own length with the forward speed $\bar{U}$. Mathematically the motion is slow, therefore, if

$$y \gg vl/\bar{U} \gg \dot{v}l^2/\bar{U}^2 \gg \ldots. \tag{8}$$

It should be noted that in this paragraph $y, v, \dot{v}, \ldots$ represent the absolute magnitudes of the variables and are all positive. Moreover, if the motion is oscillatory, then $y, v, \dot{v}, \ldots$ here denote the amplitudes of the variables.

If, therefore, all the components of the disturbed motion are small in this sense, then the Taylor series (6) can be further curtailed by omitting the terms containing the higher order derivatives $Y_{\ddot{v}}\ddot{v}, \ldots$. The component ΔY is now adequately specified 'to the first order' as

$$\Delta Y = Y_\phi \phi + Y_v v + Y_{\dot{v}} \dot{v} + Y_p p + Y_{\dot{p}} \dot{p} + Y_r r + Y_{\dot{r}} \dot{r} + Y(t), \tag{9}$$

where now we must regard $Y_\phi, Y_v, \ldots, Y_{\dot{r}}$ as 'slow motion derivatives' or, strictly, derivatives present in the slow motion approximation to ΔY. The fundamental assumption is made that the motion is so slow that only position (ϕ), velocity (v, p, r) and acceleration ($\dot{v}, \dot{p}, \dot{r}$) terms need be retained in a Taylor series expansion of the function (5)—aside, that is, from a possible time-dependent forcing term $Y(t)$.

Proper justification of this approximation is by no means easy, but the approach is now well established and a further discussion of it will not be attempted here. The approximation is important because many problems facing the analyst in ship dynamics relate to disturbed motions which are slow—as, for example, in the transition from a stable to a divergent, non-oscillatory, unstable reference motion.

When values of certain of the derivatives are needed for use in equations of motion they have usually to be measured, since theoretical methods have not proved sufficiently reliable for experiments to be dispensed with. The quantity Y_v, for example, can be found from towing tests in a long tank using a yawed model (see Abkowitz 1964). Other experimental approaches may be relevant for other derivatives but, for the sake of explanation, we shall refer only to the sway derivatives.

The derivative $Y_{\dot{v}}$ is usually large and is therefore of importance. It is, however, almost impossible to measure it directly in a towing tank without special apparatus, since such measurements would require the model to suffer a sway acceleration with no sway velocity. (A whirling arm gives centripetal acceleration without centripetal velocity, but of course with rotary motion as well. The centripetal acceleration is, moreover, only a second order effect and is thus not within the bounds of linear theory.)

This difficulty can be avoided by the use of a 'planar motion mechanism'. The technique—which appears to be full of promise—was first described by Gertler (1959), who referred particularly to its use with submarine models.† In one form of test the mechanism is adjusted to impart a sinusoidal sway displacement y to a model that is towed at the test speed $\bar{U}$ down a tank. If $y = y_0 \sin \omega t$ then

$$\left.\begin{aligned} v &= y_0 \omega \cos \omega t, \\ \dot{v} &= -y_0 \omega^2 \sin \omega t. \end{aligned}\right\} \tag{10}$$

In effect it is suggested that, during such a test, variation of the fluid force acting athwartships arises only from the terms $Y_v v$ and $Y_{\dot{v}} \dot{v}$ in equation (9) so that

$$\Delta Y = Y_v(y_0 \omega \cos \omega t) + Y_{\dot{v}}(-y_0 \omega^2 \sin \omega t). \tag{11}$$

Thus the amplitude of the measured component of ΔY that is in phase with the imposed displacement gives $-\omega^2 y_0 Y_{\dot{v}}$ and the amplitude of the observed component of ΔY that is in quadrature with y gives $\omega y_0 Y_v$.

When such tests are performed it is found that the derivatives so measured are often frequency-dependent. Since the object is to discover the values of the slow motion derivatives, we may suppose that interest should be focussed on the values to which Y_v and $Y_{\dot{v}}$ in equation (11) tend as ω is made smaller and smaller.

Leaving aside the case of a sinusoidal instability of non-negligible frequency for a moment, there are two possible motions that can be executed exactly at the boundary between stable and unstable motion. These are (*a*) slow sinusoidal motion and (*b*) slow non-oscillatory motion, these being associated respectively with the onset of an oscillatory instability and a divergent, non-oscillatory instability. In the former case inequality (8) stipulates that the frequency ω should be small (in fact $\omega \ll \bar{U}/l$); the second possibility requires that ω is exactly zero and that inequality (8) is still satisfied. While this approach appears to be quite simple, it is perhaps likely to raise questions. Indeed it is not clear theoretically that the values of Y_v and $Y_{\dot{v}}$ measured with a planar motion mechanism for $\omega \to 0$ are identical to the corresponding non-oscillatory slow motion derivatives. For the condition $\omega \to 0$ leads to a motion in which $v = 0 = \dot{v}$, whereas slow non-oscillatory motion corresponds to non-zero but small v and $\dot{v}$. As noted earlier, Y_v (though not $Y_{\dot{v}}$) can be measured directly by a non-oscillatory test in which a yawed model is towed in a long tank so that, in theory at least, this point can be clarified by comparing the results of both types of test. Indeed such tests show discrepancies between the two sets of results (Mandel 1967), though of course these differences may be due to variations in other test parameters.

Again, just as the definition of Y_v clearly relates to a hypothetical measurement that is made when $\dot{v} = 0$, so the definition of $Y_{\dot{v}}$ refers to a condition when $v = 0$ (i.e. the *partial* derivatives (7)

† The technique does not appear to be as useful for accurate measurements with aircraft models because it is difficult to prevent important extraneous motions.

are evaluated for the condition $\phi = 0 = v = \dot{v} = \ldots = \dot{r}$). Thus the planar motion method of obtaining $Y_{\dot{v}}$ differs fundamentally from the towing test determination of Y_v, since $Y_{\dot{v}}$ is not always measured directly in a manner suggested by its definition. Some reassurance can be found, however, in that it is possible in principle to derive Y_v by measuring ΔY when $\dot{v} = 0$ and likewise to determine $Y_{\dot{v}}$ from ΔY when $v = 0$ (see Abkowitz 1964). For:

$$\left.\begin{aligned} v = 0;\ \Delta Y &= -Y_{\dot{v}}(y_0\omega^2 \sin \omega t) = Y_{\dot{v}}\dot{v}, \quad \text{when} \quad t = \pi/2\omega \\ \text{and} \quad \dot{v} = 0;\ \Delta Y &= Y_v(y_0\omega \cos \omega t) = Y_v v, \quad \text{when} \quad t = \pi/\omega. \end{aligned}\right\} \tag{12}$$

There appears to be no real doubt as to the correctness of the oscillatory technique as a practical means of determining slow motion derivatives for non-oscillatory motion. But the fact that the newcomer to the technique must be forgiven if he queries its theoretical background is unfortunate since the technique is undoubtedly of vital importance and may even supplant direct measurement in the fullness of time. Moreover, very little reassurance can be found in published data.

The planar motion mechanism can serve at least two other purposes. First, it provides a means of measuring derivatives such as Y_r and N_r, so removing the need for a rotating arm facility—in theory at least. (The rotating arm mechanism imposes a non-oscillatory rate of yaw, r, whereas the planar motion device produces a sinusoidal yaw so that, as before, the distinction between oscillatory and non-oscillatory slow motion derivatives must be remembered). The planar motion mechanism also produces values of r smaller than those normally attainable with a rotating arm facility, which is advantageous in an experimental determination of Y_r and N_r. Secondly, the planar motion mechanism can be used to measure oscillatory derivatives for frequencies which are too high to permit one to use the slow motion approximation. Frequency-dependent oscillatory derivatives would be needed, for example, to estimate the onset of a general oscillatory instability. An alternative approach will therefore be described now in terms of 'oscillatory coefficients'—quantities which are potentially more useful, since they give rise to 'slow motion derivatives' as a special case.

Oscillatory coefficients

For simplicity consider a towing test in which the planar motion mechanism imposes a pure sinusoidal sway motion $y = y_0 \sin \omega t$ on a model. In these circumstances the general linear expression (6) for ΔY reduces to

$$\begin{aligned} \Delta Y &= Y_v v + Y_{\dot{v}}\dot{v} + Y_{\ddot{v}}\ddot{v} + \ldots \\ &= (Y_v - \omega^2 Y_{\ddot{v}} + \ldots)\,(y_0\omega \cos \omega t) + (Y_{\dot{v}} - \omega^2 Y_{\dddot{v}} + \ldots)\,(-y_0\omega^2 \sin \omega t). \end{aligned} \tag{13}$$

This expression can be rewritten in the form

$$\Delta Y = \tilde{Y}_v v + \tilde{Y}_{\dot{v}}\dot{v}. \tag{14}$$

The quantities $\tilde{Y}_v$ and $\tilde{Y}_{\dot{v}}$ may be referred to as 'oscillatory coefficients'. Quantities of this type are used in aeronautics. The value of $\tilde{Y}_v$ is obtained from that component of ΔY which is in quadrature with the sway displacement, while $\tilde{Y}_{\dot{v}}$ is found from the 'in-phase component'. It should be noted that $\tilde{Y}_v$ and $\tilde{Y}_{\dot{v}}$ are just coefficients multiplying v and $\dot{v}$ respectively and are not true derivatives in the sense of the definitions (7) even though they are sometimes referred to in the aeronautical literature as 'oscillatory derivatives'. Nevertheless, if ΔY were determined for a chosen frequency and for various values of y_0, then the quadrature and in-phase components could be plotted against the amplitude of v and $\dot{v}$ respectively and $\tilde{Y}_v$ and $\tilde{Y}_{\dot{v}}$ estimated from the gradients of the appropriate graphs at $y_0 = 0$ (i.e. $v = 0 = \dot{v}$).

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Similar arguments may be advanced for the dependence of ΔY on ϕ and its derivatives and on r and its derivatives, so that in all

$$\Delta Y = \tilde{Y}_\phi \phi + \tilde{Y}_p p + \tilde{Y}_v v + \tilde{Y}_{\dot v} \dot v + \tilde{Y}_r r + \tilde{Y}_{\dot r} \dot r + Y(t). \tag{15}$$

Notice that there is no oscillatory coefficient $\tilde{Y}_{\dot p}$ since it is no longer possible to distinguish between $\tilde{Y}_\phi$ and $\tilde{Y}_{\dot p}$ (both being in phase with roll). Observe too that all the oscillatory coefficients are frequency-dependent. This property is suggested by the explicit appearance of factors like $\omega^2, \omega^4 \ldots$ in the expressions for $\tilde{Y}_v, \tilde{Y}_{\dot v}$.

To return to the oscillatory coefficients $\tilde{Y}_v$ and $\tilde{Y}_{\dot v}$, it will be seen that in general no question arises now of determining one without the other. They are, so to speak, placed on the same footing. If they are measured as a pair and if the frequency is made very small, then they approximate closely to the slow motion derivatives Y_v and $Y_{\dot v}$ respectively. For example,

$$\lim_{\omega \to 0} \tilde{Y}_v = \lim_{\omega \to 0} [Y_v - \omega^2 Y_{\ddot v} + \omega^4 Y_{\ddddot v} \ldots] = Y_v. \tag{16}$$

We have seen therefore that there are at least two special types of disturbed motion for which it is possible approximately to express the incremental fluid forces and moments (such as ΔY) as linear combinations of the instantaneous displacements, velocities and accelerations (such as $\phi, v, p, r, \dot v, \dot p$ and $\dot r$). These special classes are (i) very slow motions (both oscillatory and non-oscillatory) for which we may use slow motion derivatives in equation (9), and (ii) oscillatory motions for which we may use oscillatory coefficients as in expression (15).

Fortunately one is able in this way to deal with some types of ship motions which are of interest to the analyst. In many problems the slow motion derivatives are adequate, but for some purposes the oscillatory coefficients may be required. This is the case when the ship motion is sinusoidal and of significant frequency, either as a consequence of imposed motions of control surfaces (rudders, hydroplanes or stabilizers) or alternatively as a result of operating at the boundary of an oscillatory instability. Such quantities could also be valuable in the analysis of seakeeping, where the fluid forces on a ship due to surface waves depend on frequency.

2. Theory of the planar motion mechanism in terms of oscillatory coefficients and applied to submarine models

Oscillatory motions in surge, heave or pitch

It is a familiar feature of linear systems in general that 'a sinusoidal cause will produce a sinusoidal effect having the same frequency'. This effect may only emerge as a steady state after the effects of initial conditions have died out, but emerge it will eventually if the system is stable. Suppose, then, that the planar motion mechanism imparts a sinusoidal heaving motion to a model that is towed at some constant speed $\bar{U}$ while it is submerged (figure 1). The steady reference motion about which the heaving motion takes place is

$$\mathbf{U} = \bar{U}\hat{\imath} + 0\hat{\jmath} + 0\hat{k}. \tag{17}$$

The imposed disturbance is such that

$$\left.\begin{aligned} u &= 0 = q, \\ z &= z_0 \sin \omega t. \end{aligned}\right\} \tag{18}$$

The heaving motions modify the steady fluid forces and moments by the addition of force components $\Delta X\hat{\imath}, \Delta Z\hat{k}$ and a moment $\Delta M\hat{\jmath}$. The quantities $\Delta X, \Delta Z, \Delta M$ will vary sinusoidally with the frequency ω after a steady state of motion has been reached but there is no ground for supposing either that they will be in phase with z or that they will be in phase with each other.† Consider just one of these quantities, ΔM for instance; it can be expressed in the form

$$\Delta M = A \cos \omega t + B \sin \omega t,$$

where A and B can be measured; if

$$\tilde{M}_w = A/z_0\omega, \quad \tilde{M}_z = B/z_0,$$

this may be written in the form $$\Delta M = \tilde{M}_w w + \tilde{M}_z z,$$

since $\dot{z} = z_0 \omega \cos \omega t = w$. The constants $\tilde{M}_w$ and $\tilde{M}_z$ are typical 'oscillatory coefficients'.

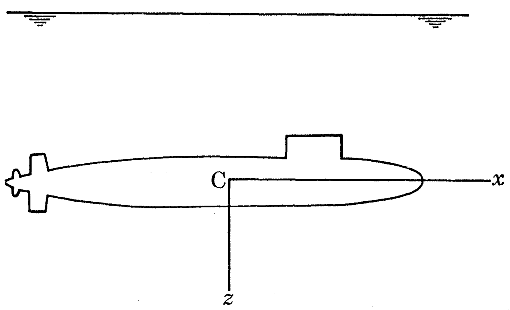

FIGURE 1

If we were studying, not sinusoidal but slow unidirectional disturbances from the steady motion, we should assert that ΔM has no direct dependence on z—only on its time derivatives. For reasons that will become obvious, then, we should prefer to write

$$\tilde{M}_{\dot{w}} = -B/z_0\omega^2,$$

whence $$\Delta M = \tilde{M}_w w + \tilde{M}_{\dot{w}} \dot{w} \tag{19}$$

since $\ddot{z} = -\omega^2 z_0 \sin \omega t = \dot{w}$.

In just the same way the two symmetric components of the fluid force can be written in the form $$\Delta X = \tilde{X}_w w + \tilde{X}_{\dot{w}} \dot{w}, \quad \Delta Z = \tilde{Z}_w w + \tilde{Z}_{\dot{w}} \dot{w}, \tag{20}$$

in which the oscillatory coefficients can be measured by test. In all three cases, the first term (that is proportional to w) represents the quadrature component while the other represents the in-phase component.

Using the same reference motion (17) we could impart a sinusoidal surge motion

$$x = x_0 \sin \omega t,$$
$$u = x_0 \omega \cos \omega t,$$
$$\dots\dots\dots\dots\dots\dots,$$

instead of the heave z. Alternatively a sinusoidal pitching motion

$$\theta = \theta_0 \sin \omega t,$$
$$q = \theta_0 \omega \cos \omega t,$$
$$\dots\dots\dots\dots\dots\dots,$$

† This is a consequence of the so-called 'Wagner Effect' that has been widely studied in aeronautics but has received almost no attention in the maritime literature.

could be imposed, although in that particular case the towing speed would only be equal to the reference speed to the first order of approximation. Yet again the hydroplane angles η_1 and η_2 could be varied sinusoidally so that

$$\eta_1 = (\eta_1)_0 \sin\omega t,$$
$$\dot{\eta}_1 = (\eta_1)_0\, \omega \cos\omega t,$$
$$\cdots\cdots\cdots\cdots,$$

or

$$\eta_2 = (\eta_2)_0 \sin\omega t,$$
$$\dot{\eta}_2 = (\eta_2)_0\, \omega \cos\omega t,$$
$$\cdots\cdots\cdots\cdots$$

For all of these imposed motions, expressions of the type (19) and (20) may be written down and the multipliers

$$\tilde{X}_u, \tilde{X}_{\dot{u}}, \ldots, (\tilde{M}_\eta)_2, (\tilde{M}_{\dot{\eta}})_2$$

found from measurements of the in-phase and quadrature components.

Suppose that the sinusoidal disturbances u, w, q, η_1, η_2 are imposed simultaneously with the same frequency. For a submerged submarine model we should obtain the expressions

$$\left.\begin{aligned}\Delta X &= \tilde{X}_u u + \tilde{X}_{\dot{u}}\dot{u} + \tilde{X}_w w + \tilde{X}_{\dot{w}}\dot{w} + \tilde{X}_q q + \tilde{X}_{\dot{q}}\dot{q} + (\tilde{X}_\eta)_1\eta_1 + (\tilde{X}_{\dot{\eta}})_1\dot{\eta}_1 + (\tilde{X}_\eta)_2\eta_2 + (\tilde{X}_{\dot{\eta}})_2\dot{\eta}_2,\\ \Delta Z &= \tilde{Z}_u u + \tilde{Z}_{\dot{u}}\dot{u} + \tilde{Z}_w w + \tilde{Z}_{\dot{w}}\dot{w} + \tilde{Z}_q q + \tilde{Z}_{\dot{q}}\dot{q} + (\tilde{Z}_\eta)_1\eta_1 + (\tilde{Z}_{\dot{\eta}})_1\dot{\eta}_1 + (\tilde{Z}_\eta)_2\eta_2 + (\tilde{Z}_{\dot{\eta}})_2\dot{\eta}_2,\\ \Delta M &= \tilde{M}_u u + \tilde{M}_{\dot{u}}\dot{u} + \tilde{M}_w w + \tilde{M}_{\dot{w}}\dot{w} + \tilde{M}_\theta\theta + \tilde{M}_q q + (\tilde{M}_\eta)_1\eta_1 + (\tilde{M}_{\dot{\eta}})_1\dot{\eta}_1 + (\tilde{M}_\eta)_2\eta_2 + (\tilde{M}_{\dot{\eta}})_2\dot{\eta}_2.\end{aligned}\right\} \quad (21)$$

Notice that, in writing these expressions we have:

(*a*) Invoked the usual assumption that the symmetric forces and moments are independent of the antisymmetric variables ϕ, v, p, r, ζ.

(*b*) Used $\tilde{M}_\theta\theta$ for the in-phase moment rather than $\tilde{M}_{\dot{q}}\dot{q}$ because, when slow motion derivatives are employed, a term $M_\theta\theta = -mgh\theta$ is introduced by the fact that the centre of buoyancy is located at a height h above the centre of mass.

(*c*) Omitted a term $\tilde{M}_{\dot{q}}\dot{q}$ comparable with $M_{\dot{q}}\dot{q}$ in the 'slow motion equations' because one can only discriminate between the in-phase and quadrature so that $\tilde{M}_{\dot{q}}\dot{q}$ is indistinguishable from $\tilde{M}_\theta\theta$.

Expressions like those of equations (21) are to be used in the equations of motion governing symmetric disturbances, namely

$$\Delta X - mg\theta = m\dot{u}, \quad \Delta Z = m(\dot{w} - q\bar{U}), \quad \Delta M = I_y\dot{q}. \quad (22)$$

But it must be remembered that if equations (21) are relevant, all the disturbances are sinusoidal and of the same frequency ω.

It will be appreciated that vertical symmetric oscillations of a submerged submarine have been chosen for the purposes of explanation. Similar arguments apply to antisymmetric disturbances which, for a submerged submarine model, may be imposed by mounting the model on its side and oscillating it in the vertical plane. It is also possible to impart a sinusoidal rolling motion to the model by suitably adapting the mechanism; but again this requires no fundamental modification of the underlying theory.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Some properties of the measured oscillatory coefficients

By means of dimensional analysis it can be shown that, for example,

$$\tilde{M}'_w = \frac{\tilde{M}_w}{\frac{1}{2}\rho\bar{U}l^3} = f\left(\text{Fn}, \text{Re}, \frac{\omega l}{\bar{U}}\right). \tag{23}$$

That is to say a 'non-dimensionalized' oscillatory coefficient depends (for a given shape of model) on: Fn, the Froude number, Re, the Reynolds number, $\nu = \omega l/\bar{U}$, the dimensionless 'reduced frequency'. In general, then, the constants that we have called 'oscillatory coefficients' are frequency-dependent. In this respect they differ in a significant manner from ordinary slow motion derivatives. Unfortunately, no data have yet been published for a submarine showing this variation of a typical oscillatory coefficient with frequency. It is to be expected, however, that as the frequency is made very small the oscillatory coefficient becomes equal to the more familiar slow motion derivative. Thus we should expect that

$$\lim_{\omega\to 0} \tilde{M}_w = M_w, \quad \lim_{\omega\to 0} \tilde{M}_{\dot{w}} = M_{\dot{w}} \tag{24}$$

as has already been mentioned.

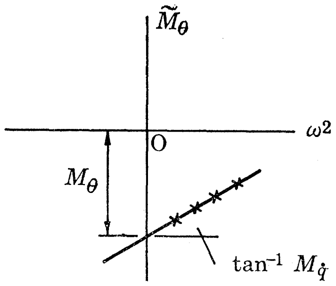

FIGURE 2

Suppose that it is necessary to find the slow motion derivatives M_θ, M_q, $M_{\dot{q}}$ for use in the third of equations (22). It would appear that only M_θ amd M_q can be found, $M_{\dot{q}}$ being inseparable from M_θ. By analogy with equation (13) the oscillatory coefficient $\tilde{M}_\theta$ is a quantity of the form

$$\tilde{M}_\theta = M_\theta - \omega^2 M_{\dot{q}} + \omega^4 M_{\dddot{q}} - \dots. \tag{25}$$

If therefore the pitch $\theta = \theta_0 \sin \omega t$ is imposed and a curve is plotted of the in-phase component $\tilde{M}_\theta$ against ω^2 one may seek to find both M_θ and $M_{\dot{q}}$ using the technique indicated by the sketch in figure 2. In as much as it is usual to take $M_\theta = -mgh$ in practice, one should also be able to apply a useful check on this part of the result.

Technique of measurement

It is usual in the study of ship and submarine dynamics to separate the analysis of surging motions u from that of the remaining symmetric motions w and q. We shall therefore focus our attention on the latter. We shall examine briefly (a) the method by which the desired sinusoidal motions are imparted to a model, and (b) the way in which the oscillatory coefficients may be calculated from the measured data.

The oscillatory coefficients may be measured conveniently in a towing tank by the use of a planar motion mechanism which imparts a known vertical displacement at each of two points

P and Q, of a model. These are best chosen on the centre line of the model and at equal distances l_0 say, fore and aft of the centre of mass C (figure 3).† The displacements, which will be denoted by z_1 and z_2, are usually made to vary sinusoidally by slider crank mechanisms or by Scotch yokes. (Alternatives do suggest themselves but it would be out of place to discuss them here.) It must be remembered that the displacements z_1 and z_2 are vertical and so not necessarily parallel to the moving axis Cz.

The forces that have to be applied to the model to maintain the sinusoidal motion are measured at the two points P and Q, there the planar motion mechanism is attached. To be more exact, it is the components of these forces in the Cz direction which are measured; they will be denoted by F_1 and F_2. The planar motion mechanism is mounted on the carriage of the towing tank so that the model may be given a velocity $\bar{U}$ along the centre line of the tank together with a harmonic vertical motion relative to the carriage. We shall assume that the towing speed may be treated as the reference speed, any fluctuation of the speed in the direction Cx being negligible.

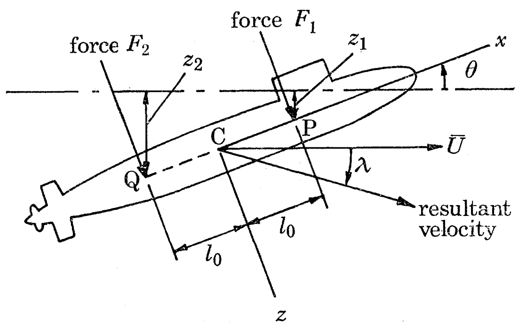

FIGURE 3

Heave coefficients of a submerged submarine model

When the coefficients $\tilde{Z}_w$, $\tilde{Z}_{\dot{w}}$, $\tilde{M}_w$, $\tilde{M}_{\dot{w}}$ are measured, the model is made to undergo a pure heaving motion whilst it is towed at constant speed along the tank. The planar motion mechanism is adjusted so that z_1 and z_2 have the same amplitude and are in phase, being given by $z = z_0 \sin \omega t$ so that

$$w = z_0 \omega \cos \omega t, \quad \dot{w} = -z_0 \omega^2 \sin \omega t. \tag{26}$$

The displacements z_1 and z_2 are now parallel to Cz, which remains vertical. The steady state amplitudes and phases of the forces F_1 and F_2 are recorded; suppose they are given by

$$\left.\begin{aligned} F_1 &= F_{1*} + (F_1)_{\text{in}} \sin \omega t + (F_1)_{\text{out}} \cos \omega t, \\ F_2 &= F_{2*} + (F_2)_{\text{in}} \sin \omega t + (F_2)_{\text{out}} \cos \omega t, \end{aligned}\right\} \tag{27}$$

where, it will be noted, allowance has been made for constant as well as fluctuating components. The oscillatory coefficients for heave may now be found from these measurements, but their derivation requires the use of equations of motion which are not only simplified relationships but have been simplified in more ways than one in the literature.

Remembering that we are concerned with sinusoidal heaving motions of a submerged submarine, consider the heave equation of motion. It is the second of equations (22) with the expression for ΔZ that is given in equations (21). That is

$$-\tilde{Z}_u u - \tilde{Z}_{\dot{u}} \dot{u} - \tilde{Z}_w w + (m - \tilde{Z}_{\dot{w}}) \dot{w} - (\tilde{Z}_q + m\bar{U}) q - \tilde{Z}_{\dot{q}} \dot{q}$$
$$= Z_* + Z(t) + (\tilde{Z}_\eta)_1 \eta_1 + (\tilde{Z}_{\dot{\eta}})_1 \dot{\eta}_1 + (\tilde{Z}_\eta)_2 \eta_2 + (\tilde{Z}_{\dot{\eta}})_2 \dot{\eta}_2. \tag{28}$$

† In order to prevent interference of the flow round the fin by the supporting struts, the model is usually held upside down.

Notice that we have here included two extra terms —Z_* and $Z(t)$. The first of these, Z_*, is a constant for any given reference speed $\bar{U}$; it represents the dependence on the speed $\bar{U}$ of the normal force at zero angle of attack. This dependence exists because the plane Cxy is not a plane of symmetry, if only because of the presence of the fin, and it would normally be counteracted by adjustment of the zero-settings of the hydroplanes. The other force, $Z(t)$, is the imposed sinusoidal force which causes the harmonic displacements and which requires us to use oscillatory coefficients (as opposed to slow motion derivatives) in the equation.

The planar motion mechanism imparts a sinusoidal motion of pure heaving while the hydroplanes are held fixed so that equation (28) becomes

$$-\tilde{Z}_w w + (m - \tilde{Z}_{\dot{w}})\, \dot{w} = Z_* + F_* + F_{\text{in}} \sin \omega t + F_{\text{out}} \cos \omega t, \tag{29}$$

where $$F_* = F_{1*} + F_{2*}, \quad F_{\text{in}} = (F_1)_{\text{in}} + (F_2)_{\text{in}}, \quad F_{\text{out}} = (F_1)_{\text{out}} + (F_2)_{\text{out}}. \tag{30}$$

According to equation (26) we have

$$-\tilde{Z}_w z_0 \omega \cos \omega t - (m - \tilde{Z}_{\dot{w}})\, z_0 \omega^2 \sin \omega t = F_* + F_{\text{in}} \sin \omega t + F_{\text{out}} \cos \omega t, \tag{31}$$

so that $$Z_* = -F_*, \quad \tilde{Z}_w = -F_{\text{out}}/z_0 \omega, \quad \tilde{Z}_{\dot{w}} = m + F_{\text{in}}/z_0 \omega^2. \tag{32}$$

Thus we can find Z_*, $\tilde{Z}_w$ and $\tilde{Z}_{\dot{w}}$ from readings obtained for a given speed.

The pitch equation (22) is

$$-\tilde{M}_u u - \tilde{M}_{\dot{u}} \dot{u} - \tilde{M}_w w - \tilde{M}_{\dot{w}} \dot{w} - \tilde{M}_\theta \theta - \tilde{M}_q q + I_y \dot{q}$$
$$= M_* + M(t) + (\tilde{M}_\eta)_1 \eta_1 + (\tilde{M}_{\dot{\eta}})_1 \dot{\eta}_1 + (\tilde{M}_\eta)_2 \eta_2 + (\tilde{M}_{\dot{\eta}})_2 \dot{\eta}_2. \tag{33}$$

For the motion (26), therefore,

$$-\tilde{M}_w z_0 \omega \cos \omega t + \tilde{M}_{\dot{w}} z_0 \omega^2 \sin \omega t = M_* + G_* + G_{\text{in}} \sin \omega t + G_{\text{out}} \cos \omega t, \tag{34}$$

where

$$G_* = -l_0(F_{1*} - F_{2*}), \quad G_{\text{in}} = -l_0[(F_1)_{\text{in}} - (F_2)_{\text{in}}], \quad G_{\text{out}} = -l_0[(F_1)_{\text{out}} - (F_2)_{\text{out}}]. \tag{35}$$

Thus $$M_* = -G_*, \quad \tilde{M}_w = -G_{\text{out}}/z_0 \omega, \quad \tilde{M}_{\dot{w}} = G_{\text{in}}/z_0 \omega^2. \tag{36}$$

Pitch coefficients of a submerged submarine model

The model may be given a pure pitching motion while it is moving along the tank at constant speed. The model moves along a path like that shown in figure 4. The angle made by the axis Cx of the model with the horizontal is

$$\theta = \sin^{-1}\{(z_2 - z_1)/2l_0\}. \tag{37}$$

The vertical downward velocity of C is

$$\mathrm{d}[\tfrac{1}{2}(z_1 + z_2)]/\mathrm{d}t,$$

so that the resultant velocity makes an angle

$$\lambda = \tan^{-1}\left[\frac{\mathrm{d}[\frac{1}{2}(z_1 + z_2)]/\mathrm{d}t}{\bar{U}}\right], \tag{38}$$

with the horizontal. If the motion is to be one of pure pitching, it is necessary that $w = 0 = \dot{w}$ so that the velocity of C must always be tangential to the path of C. That is $\theta = -\lambda$, so that for small angles

$$-\left(\frac{z_2 - z_1}{2l_0}\right) \doteqdot \frac{\mathrm{d}[\frac{1}{2}(z_1 + z_2)]/\mathrm{d}t}{\bar{U}}. \tag{39}$$

To produce a pure pitching motion, then, z_1 and z_2 must satisfy this relationship. It is easy to show that this requirement is met if z_1 and z_2 have the form

$$z_1 = z_0 \cos(\omega t + \tfrac{1}{2}\epsilon), \quad z_2 = z_0 \cos(\omega t - \tfrac{1}{2}\epsilon), \tag{40}$$

where the phase lead of z_1 with respect to z_2 is

$$\epsilon = 2 \tan^{-1}(\omega l_0/\bar{U}). \tag{41}$$

That is to say, a pure pitching motion is obtained if the amplitudes of z_1 and z_2 are the same and the phase difference between them depends upon the frequency of oscillation and the towing speed in this way; z_1 must lead z_2 by the angle† ϵ.

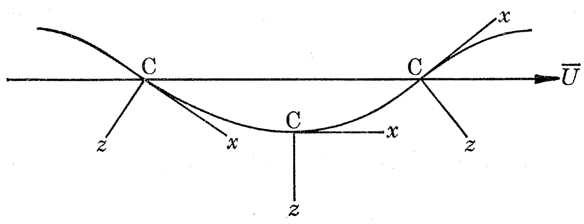

FIGURE 4

If the displacements z_1 and z_2 are of the form given in equation (40), they are no longer parallel to Cz, as the direction of the latter varies. The angle of pitch is given by

$$\sin\theta = \frac{z_0[\cos(\omega t - \tfrac{1}{2}\epsilon) - \cos(\omega t + \tfrac{1}{2}\epsilon)]}{2l_0}$$

according to equation (37). This may be simplified to

$$\sin\theta = (z_0/l_0) \sin\omega t \sin\tfrac{1}{2}\epsilon,$$

so that, for small angles of pitch, we may write

$$\theta = \theta_0 \sin\omega t, \tag{42}$$

where $$\theta_0 = (z_0/l_0)\sin\tfrac{1}{2}\epsilon. \tag{43}$$

It follows that $$q = \dot{\theta} = \theta_0\omega\cos\omega t, \quad \dot{q} = \ddot{\theta} = -\theta_0\omega^2\sin\omega t. \tag{44}$$

If the steady state force amplitudes F_1 and F_2 are measured and have the form (27) it is possible to calculate the pitch coefficients. Notice that having arranged z_1 and z_2 to give a pure pitching motion (by adjustment of ϵ, their difference of phase) we can conveniently refer the phase of the applied forces F_1 and F_2 to θ, the angle of pitch—rather than to the dummy variable with reference to which z_1 and z_2 are set.

The heave equation (28) now becomes

$$-(\tilde{Z}_q + m\bar{U})(\theta_0\omega\cos\omega t) + \tilde{Z}_{\dot{q}}(\theta_0\omega^2\sin\omega t) = Z_* + F_* + F_{\text{in}}\sin\omega t + F_{\text{out}}\cos\omega t, \tag{45}$$

so that $$Z_* = -F_*, \quad \tilde{Z}_q = -m\bar{U} - F_{\text{out}}/\theta_0\omega, \quad \tilde{Z}_{\dot{q}} = F_{\text{in}}/\theta_0\omega^2. \tag{46}$$

The pitch equation (33) is now

$$-\tilde{M}_\theta\theta_0\sin\omega t - \tilde{M}_q\theta_0\omega\cos\omega t - I_y\theta_0\omega^2\sin\omega t = M_* + G_* + G_{\text{in}}\sin\omega t + G_{\text{out}}\cos\omega t, \tag{47}$$

† An alternative expression for ϵ is commonly used in the literature; it is

$$\epsilon = \cos^{-1}\{(1-\mu^2)/(1+\mu^2)\} \quad \text{where} \quad \mu = \omega l_0/\bar{U}.$$

whence $$M_* = -G_*, \quad \tilde{M}_\theta = -I_y\omega^2 - G_{\text{in}}/\theta_0, \quad \tilde{M}_q = -G_{\text{out}}/\theta_0\omega. \tag{48}$$

And, as we have already mentioned, the oscillatory coefficient $\tilde{M}_\theta$ can be made to yield both of the slow motion derivatives M_θ and $M_{\dot{q}}$.

Oscillatory coefficients associated with control surfaces

It has already been indicated that the idea of oscillatory coefficients is applicable to control surface deflexions as well as to wholesale vertical symmetric motions of a submerged submarine. The vertical displacements z_1 and z_2 are now held fixed while the model is towed down the tank. (Notice that, by adjusting z_1 and z_2 before runs one can investigate the variation of Z_* and M_* for different orientations of the axes Cxyz that are fixed in the model.)

Suppose that it is wished to measure $(\tilde{Z}_\eta)_1$, $(\tilde{Z}_{\dot{\eta}})_1$, $(\tilde{M}_\eta)_1$ and $(\tilde{M}_{\dot{\eta}})_1$ for the forward hydroplane.† With z_1 and z_2 fixed the model is towed along the tank while η_1 is made to vary sinusoidally so that

$$\eta_1 = \eta_{10}\sin\omega t, \quad \dot{\eta}_1 = \eta_{10}\,\omega\cos\omega t. \tag{49}$$

The transducers measure F_1 and F_2 as before and the heave equation (28) now becomes

$$0 = Z_* + F_* + F_{\text{in}}\sin\omega t + F_{\text{out}}\cos\omega t + (\tilde{Z}_\eta)_1(\eta_{10}\sin\omega t) + (\tilde{Z}_{\dot{\eta}})_1(\eta_{10}\,\omega\cos\omega t), \tag{50}$$

whence $$Z_* = -F_*, \quad (\tilde{Z}_\eta)_1 = -F_{\text{in}}/\eta_{10}, \quad (\tilde{Z}_{\dot{\eta}})_1 = -F_{\text{out}}/\eta_{10}\,\omega. \tag{51}$$

The pitch equation (33) now becomes

$$0 = M_* + G_* + G_{\text{in}}\sin\omega t + G_{\text{out}}\cos\omega t + (\tilde{M}_\eta)_1\eta_{10}\sin\omega t + (\tilde{M}_{\dot{\eta}})_1\eta_{10}\,\omega\cos\omega t, \tag{52}$$

so that $$M_* = -G_*, \quad (\tilde{M}_\eta)_1 = -G_{\text{in}}/\eta_{10}, \quad (\tilde{M}_{\dot{\eta}})_1 = -G_{\text{out}}/\eta_{10}\,\omega. \tag{53}$$

Oscillatory coefficients of a submarine in roll

The planar motion mechanism may be capable of giving a submarine model a sinusoidal rolling oscillation. The main support struts attached to the model at the points P and Q in figure 3 are locked, so that the model is only free to move in roll, as it is towed at some constant speed $\bar{U}$. Starting from the steady reference motion $\boldsymbol{U} = \bar{U}\hat{\imath}$, therefore, the imposed disturbance is such that

$$\phi = \phi_0\sin\omega t, \quad u = 0 = v = w = q = r. \tag{54}$$

Since ϕ varies p and $\dot{p}$ are also non-zero.

Due to the symmetry of the submarine about the Cxz plane the linear theory predicts that the disturbed motion (54) will generate additional fluid forces and moments $\Delta Y\hat{\jmath}$, $\Delta L\hat{\imath}$ and $\Delta N\hat{k}$, whereas $\Delta X\hat{\imath}$, $\Delta Z\hat{k}$ and $\Delta M\hat{\imath}$ will be zero to the first order of the small variable ϕ. If the imposed motion (54) is sinusoidal and of frequency ω. these incremental forces and moments can be expressed in terms of oscillatory coefficients which are functions of ω in the form

$$\Delta Y = \tilde{Y}_\phi\phi + \tilde{Y}_p p, \quad \Delta L = \tilde{L}_\phi\phi + \tilde{L}_p p, \quad \Delta N = \tilde{N}_p p + \tilde{N}_{\dot{p}}\dot{p}. \tag{55}$$

It should be noted that in writing expressions (55) allowance has been made for the fact that for a submerged vessel ΔY and ΔL depend on ϕ directly, as well as $p, \dot{p}, \ddot{p}, \ldots$, whereas ΔN is independent of ϕ. This difference is caused by the buoyancy forces acting on the submerged body.‡ For a

† A similar analysis would be needed for the after hydroplane.

‡ It is customary in tests with a planar motion mechanism to ensure that the centre of buoyancy of the model lies on the Cz axis. If the centre of buoyancy were displaced from this axis in the plane of symmetry Cxz, then ΔN would also be directly dependent on ϕ.

surface ship and a submarine on the surface, of course, ΔY, ΔL and ΔN all depend on ϕ, as in these circumstances the whole flow pattern is altered by imposing a roll displacement ϕ.

The linear equations describing the motion following any anti-symmetric disturbance have the form

$$\Delta Y + Y(t) + mg\phi = m(\dot{v} + r\bar{U}), \quad \Delta L + L(t) = I_x\dot{p} - I_{xz}\dot{r}, \quad \Delta N + N(t) = I_z\dot{r} - I_{zx}\dot{p}, \tag{56}$$

where the terms $Y(t)$, $L(t)$ and $N(t)$ represent possible external applied loadings. For the motion described by equations (54) these terms are associated with a torque T applied to the model by the rolling mechanism together with transverse forces P_1 and P_2 applied to the model by the struts at the points P and Q respectively (see figure 5). The notation P_1 and P_2 has been adopted here in preference to F_1 and F_2, as we are concerned with forces that are transverse rather than approximately parallel to the support struts. The torque T is detected by a suitable roll gauge† mounted in the model and the forces P_1 and P_2 are measured by modular force gauges at the points P and Q respectively. The force gauges are oriented, so that they respond only to transverse forces parallel to Cy. Strictly P_1 and P_2 are the components parallel to Cy of the total transverse forces at P and Q.

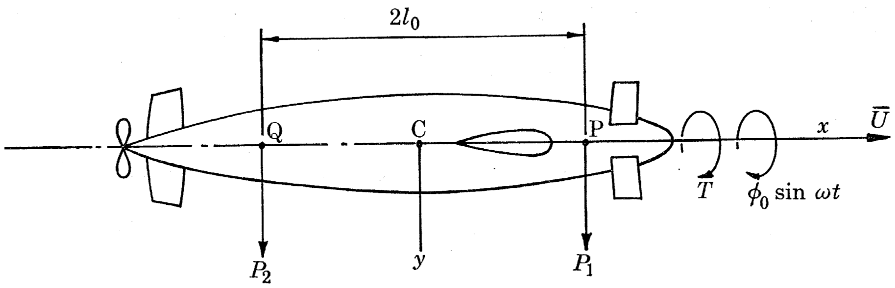

FIGURE 5

If the imposed motion is the rolling oscillation defined by equations (54), then the imposed loading is

$$\left.\begin{aligned} T(t) &= T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t, \\ P_1(t) &= (P_1)_{\text{in}} \sin \omega t + (P_1)_{\text{out}} \cos \omega t, \\ P_2(t) &= (P_2)_{\text{in}} \sin \omega t + (P_2)_{\text{out}} \cos \omega t, \end{aligned}\right\} \tag{57}$$

so that, in the equations of motion (56),

$$L(t) = T(t), \quad Y(t) = P_1(t) + P_2(t), \quad N(t) = l_0(P_1 - P_2). \tag{58}$$

Since $r = 0 = v$, $\phi = \phi_0 \sin \omega t$, we have

$$\left.\begin{aligned} \tilde{Y}_\phi \phi_0 \sin \omega t + \tilde{Y}_p \omega\phi_0 \cos \omega t + P_{\text{in}} \sin \omega t + P_{\text{out}} \cos \omega t + mg\phi_0 \sin \omega t &= 0, \\ \tilde{L}_\phi \phi_0 \sin \omega t + \tilde{L}_p \omega\phi_0 \cos \omega t + T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t &= -I_x \omega^2\phi_0 \sin \omega t, \\ \tilde{N}_p \omega\phi_0 \cos \omega t - \tilde{N}_{\dot{p}} \omega^2\phi_0 \sin \omega t + H_{\text{in}} \sin \omega t + H_{\text{out}} \cos \omega t &= I_{zx} \omega^2\phi_0 \sin \omega t, \end{aligned}\right\} \tag{59}$$

where

$$\left.\begin{aligned} P_{\text{in}} &= (P_1)_{\text{in}} + (P_2)_{\text{in}}, & P_{\text{out}} &= (P_1)_{\text{out}} + (P_2)_{\text{out}}, \\ H_{\text{in}} &= l_0[(P_1)_{\text{in}} - (P_2)_{\text{in}}], & H_{\text{out}} &= l_0[(P_1)_{\text{out}} - (P_2)_{\text{out}}]. \end{aligned}\right\} \tag{60}$$

† Some planar motion mechanisms incorporate two roll gauges in which case the applied rolling moment is the vector sum of the moments measured at these gauges (Gertler 1959). When this is the case the modifications to the results presented here are simple to make.

Separation of the in-phase and quadrature components in equations (59) produces the following expressions for the oscillatory coefficients:

$$\tilde{Y}_\phi = -mg - P_{\text{in}}/\phi_0, \quad \tilde{Y}_p = -P_{\text{out}}/\phi_0\omega, \tag{61}$$

$$\tilde{L}_\phi = -I_x\omega^2 - T_{\text{in}}/\phi_0, \quad \tilde{L}_p = -T_{\text{out}}/\phi_0\omega, \tag{62}$$

$$\tilde{N}_p = -H_{\text{out}}/\phi_0\omega, \quad \tilde{N}_{\dot{p}} = -I_{zx} + H_{\text{in}}/\phi_0\omega^2. \tag{63}$$

The corresponding expressions for surface ships can be derived in a similar manner, when allowance is made for the existence of a $\tilde{N}_\phi$ coefficient. The various oscillatory coefficients are listed in the table at the end of this paper.

In analysing antisymmetric motion the coupling between sway and yaw on the one hand (i.e. 'planar motion') and roll on the other is often assumed to be negligible (Abkowitz 1964). The validity of this assumption depends, in part, on the magnitudes of the quantities calculated from results (61)–(63) and it can thus be partially confirmed by the forced rolling tests outlined in this section.

3. Application of the theory to surface ship models

Oscillatory coefficients of surface ships

The oscillatory coefficients of surface ships (or, rather models of surface ships) can be measured by means of a planar motion mechanism. In discussing this aspect, we shall confine our attention to *horizontal* oscillatory motions so that the mechanism is now such that known lateral sinusoidal

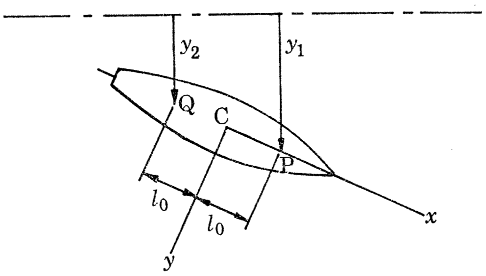

Figure 6

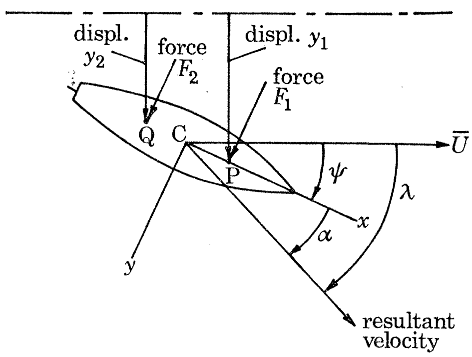

Figure 7

displacements y_1 and y_2 are imparted at each of the two points P and Q of the modal (figure 6). Notice that y_1 and y_2 are perpendicular to the direction of the reference motion and are not necessarily parallel to Cy. Descriptions of apparatus adapted for this purpose are given, for example, by Paulling & Wood (1962) and by van Leeuwen (1964). The model is towed along the centre line of the tank with a velocity $\overline{U}$ which we shall again assume to be sensibly constant and equal to the reference speed for which derivatives are to be measured. Let the components parallel to Cy of the forces applied at P and Q be denoted by F_1 and F_2 respectively (figure 7).

Sway coefficients

Suppose that the model is made to perform a pure 'swaying' motion whilst it is towed at constant speed along the tank. The mechanism is so arranged that y_1 and y_2 have the same amplitude and are in phase, both being given by $y = y_0 \sin \omega t$ so that

$$v = y_0\omega \cos \omega t, \quad \dot{v} = -y_0\omega^2 \sin \omega t. \tag{64}$$

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The motion is now such that y_1 and y_2 are parallel to Cy at all instants. The steady state amplitudes and phases of the forces F_1 and F_2 are recorded; suppose they lead to the results

$$\left.\begin{aligned} F_1 &= (F_1)_{\text{in}} \sin \omega t + (F_1)_{\text{out}} \cos \omega t, \\ F_2 &= (F_2)_{\text{in}} \sin \omega t + (F_2)_{\text{out}} \cos \omega t. \end{aligned}\right\} \tag{65}$$

The coefficients $\tilde{Y}_v$ and $\tilde{Y}_{\dot{v}}$ may be obtained from the appropriate equation of motion in a horizontal plane (such that Cxy remains horizontal) with a forcing term $Y(t)$ introduced on the right hand side. Thus we have, according to the first of equations (56),

$$-(m-\tilde{Y}_{\dot{v}})\,(y_0\,\omega^2 \sin \omega t) - \tilde{Y}_v y_0\,\omega \cos \omega t = [(F_1)_{\text{in}} + (F_2)_{\text{in}}] \sin \omega t + [(F_1)_{\text{out}} + (F_2)_{\text{out}}] \cos \omega t. \tag{66}$$

If the sine and cosine terms are now separately equated to zero, and if

$$F_{\text{in}} = (F_1)_{\text{in}} + (F_2)_{\text{in}}, \quad F_{\text{out}} = (F_1)_{\text{out}} + (F_2)_{\text{out}}, \tag{67}$$

then

$$\tilde{Y}_{\dot{v}} = m + F_{\text{in}}/y_0\,\omega^2, \quad \tilde{Y}_v = -F_{\text{out}}/y_0\,\omega. \tag{68}$$

In a similar way, the customarily used yaw equation of planar motion, with allowance made for an imposed yawing moment $N(t)$ now gives

$$\tilde{N}_{\dot{v}} y_0\,\omega^2 \sin \omega t - \tilde{N}_v y_0\,\omega \cos \omega t = l_0\{[(F_1)_{\text{in}} \sin \omega t + (F_1)_{\text{out}} \cos \omega t] - [(F_2)_{\text{in}} \sin \omega t + (F_2)_{\text{out}} \cos \omega t]\}. \tag{69}$$

If

$$G_{\text{in}} = l_0[(F_1)_{\text{in}} - (F_2)_{\text{in}}], \quad G_{\text{out}} = l_0[(F_1)_{\text{out}} - (F_2)_{\text{out}}] \tag{70}$$

we now find that

$$\tilde{N}_{\dot{v}} = G_{\text{in}}/y_0\,\omega^2, \quad \tilde{N}_v = -G_{\text{out}}/y_0\,\omega. \tag{71}$$

Thus if the in-phase and quadrature components of F_1 and F_2 are measured it is possible to determine the four sway coefficients.

Similarly, if the corresponding components of the torque $T(t)$ are observed by means of the roll gauge† (see first of equations (57)), it is possible to calculate the coefficients $\tilde{L}_v$ and $\tilde{L}_{\dot{v}}$. Thus, substituting

$$\Delta L = \tilde{L}_v v + \tilde{L}_{\dot{v}} \dot{v}, \quad L(t) = T(t) = T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t, \quad p = 0 = r, \tag{72}$$

into the second of equations (56) we find that

$$\tilde{L}_v y_0\,\omega \cos \omega t - \tilde{L}_{\dot{v}} y_0\,\omega^2 \sin \omega t + T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t = 0.$$

That is

$$\tilde{L}_v = -T_{\text{out}}/y_0\,\omega, \quad \tilde{L}_{\dot{v}} = T_{\text{in}}/y_0\,\omega^2. \tag{73}$$

It should be remembered that we need not consider a coefficient of the form $\tilde{L}_y$ as none of the fluid reactions is directly dependent on a sway displacement y.

Some of these results may be illustrated with data found by van Leeuwen (1964) for a model with:

length between perpendiculars, l	2.258 m
breadth	0.323 m
towing speed, $\overline{U}$	0.928 m s^{-1}
propeller speed	10.4 rev s^{-1}
block coefficient	0.70

† The forced rolling mechanism is, of course, locked during tests in sway and yaw, so that there is now no freedom in roll.

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Figure 8 (*a*) and (*b*) show two of the curves he found for the model with its rudder undeflected and with its propeller rotating.† They well illustrate the following points:

(i) It is possible to determine slow motion derivatives by examining the limit as $\omega \to 0$. Thus

$$(m - Y_{\dot{v}})^{\frac{1}{2}} = (\text{slope at origin in figure 8}\,(a)),$$

$$Y_v = -(\text{slope at origin in figure 8}\,(b)).$$

Moreover, the slopes of the curves, and hence the coefficients, appear to be constant in this small frequency range.

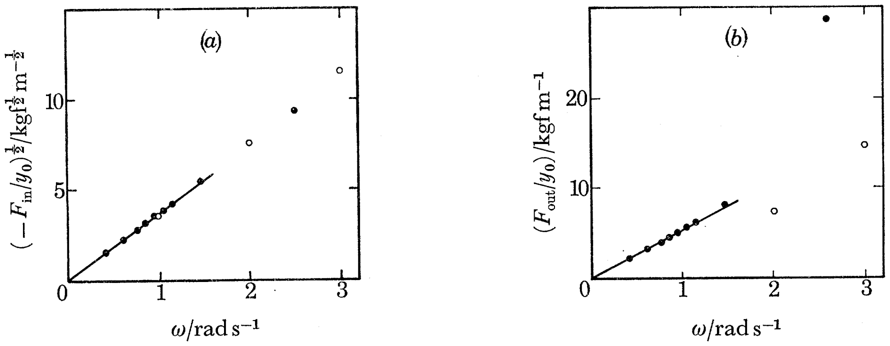

FIGURE 8. •, $y_0 = 2.5$ cm; ○, $y_0 = 10.0$ cm.

(ii) Furthermore, we can estimate the non-dimensional slow motion derivatives in this way. Very approximately,‡ the curves indicate that

$$m' - Y'_{\dot{v}} = \frac{m - Y_{\dot{v}}}{\frac{1}{2}\rho l^3} \doteqdot \frac{3.5^2 \times 9.81}{\frac{1}{2} \times 1000 \times 2.258^3} \doteqdot 0.02,$$

$$Y'_v = \frac{Y_v}{\frac{1}{2}\rho \overline{U}\, l^2} \doteqdot \frac{-5.5 \times 9.81}{\frac{1}{2} \times 1000 \times 0.928 \times 2.258^2} \doteqdot -0.02,$$

corresponding to the Froude number:

$$\text{Fn} = \frac{\overline{U}}{\surd(gl)} = \frac{0.928}{\surd(9.81 \times 2.258)} \doteqdot 0.2.$$

(iii) The value of y_0 is not important at these low frequencies since the force amplitudes are proportional to the displacement amplitudes for a given frequency.§ (This is less obvious in figure 8 (*b*) than it is in figure 8 (*a*), however.)

It is of interest to examine some of the curves of the oscillatory coefficients that van Leeuwen found. Those for Fn = 0.2 are given in figure 9 (*a*)–(*d*). The non-dimensional frequency ω' used in these curves is reckoned in the form

$$\omega' = \omega\surd(l/g) = \omega(2.258/9.81)^{\frac{1}{2}} = 0.48\omega$$

† Figures 8 to 10 have been taken from van Leeuwen's paper and have been redrawn and adapted to our present needs.

‡ Van Leeuwen gives the much more accurate figures of 0.0229 and −0.0222.

§ Van Leeuwen gives some rough limits within which this rule is reasonably accurate. For his model it was necessary that the maximum drift angle α should not exceed 10° and the amplitude of the non-dimensional yawing component r' of the angular velocity should not exceed 0.3.

where ω is in radians per second. If, instead, the 'reduced' frequency

$$\nu = \frac{\omega l}{\overline{U}} = \frac{\omega \times 2.258}{0.928} = 2.44\omega$$

is used, where again ω is in radians per second, it is perhaps easier to see how sensitively dependent the oscillatory coefficients are on frequency. Thus figure 9 (b) shows that $\tilde{N}_{\dot{v}}$ changes sign when $\omega \doteqdot 2.5$ rad/s, or $\nu \doteqdot 6.1$ rad; that is to say $\tilde{N}_{\dot{v}}$ changes sign when $6.1/2\pi$, or 0.97 cycles of sway are performed during the time the model is towed a distance equal to its own length.

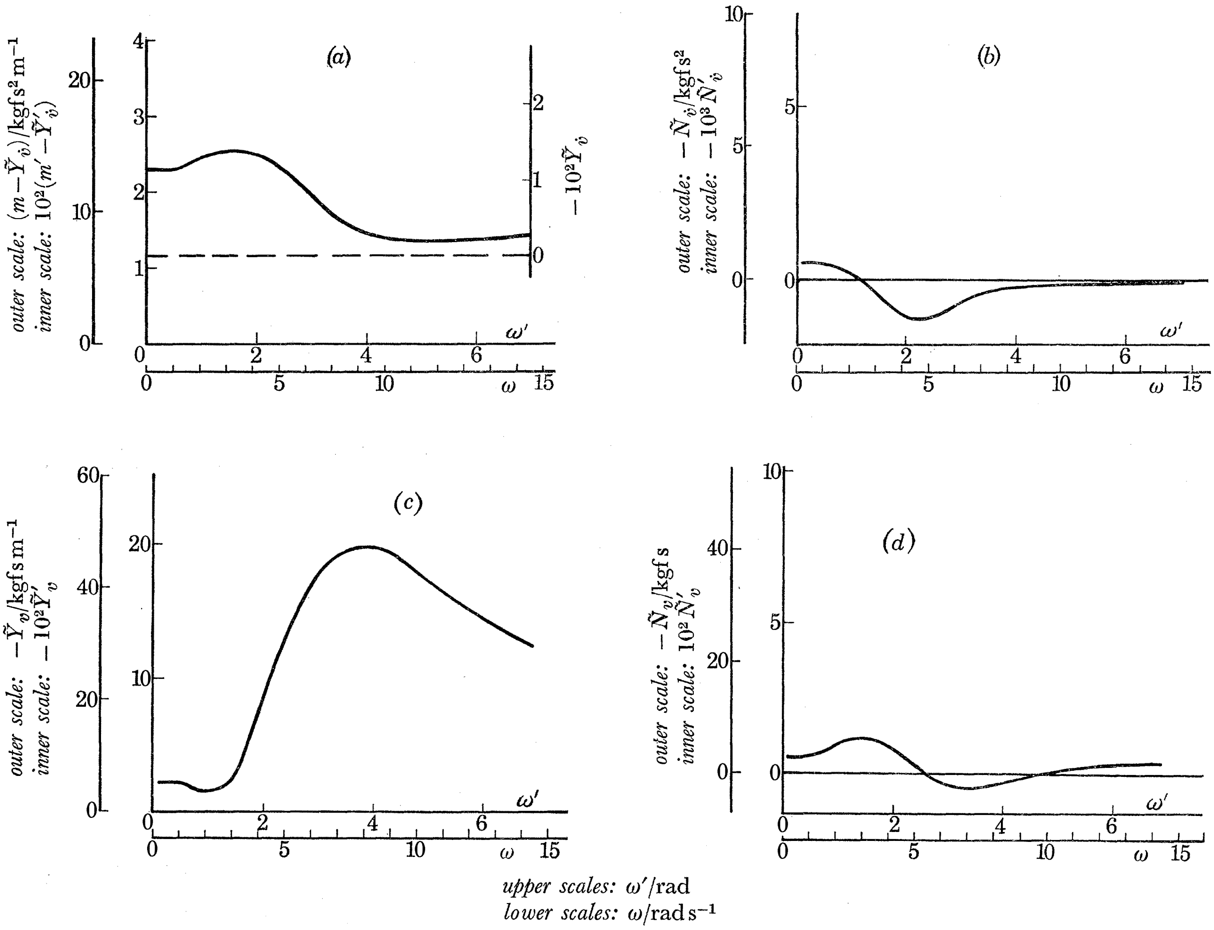

FIGURE 9

Yaw coefficients

The model performs a pure yawing motion if y_1 and y_2 are given by

$$y_1 = y_0 \cos(\omega t + \tfrac{1}{2}\epsilon), \quad y_2 = y_0 \cos(\omega t - \tfrac{1}{2}\epsilon), \tag{74}$$

where
$$\epsilon = 2 \tan^{-1}(\omega l_0/\overline{U}). \tag{75}$$

The proof of this is, of course, analogous to that relating to pure pitching oscillation of a submerged submarine model, except that $\lambda = \psi$ (see figure 7) instead of $\lambda = -\theta$. Now the direction of the body axis Cy varies, whereas y_1 and y_2 remain perpendicular to the direction of the reference motion along the tank. If the displacements y_1 and y_2 are of this form, the angle of yaw ψ is given by

$$\psi = \sin^{-1}\left(\frac{y_1-y_2}{2l_0}\right) = \sin^{-1}\left\{\frac{y_0[\cos(\omega t+\frac{1}{2}\epsilon)-\cos(\omega t-\frac{1}{2}\epsilon)]}{2l_0}\right\}$$

which may be simplified to
$$\psi \doteqdot -(y_0/l_0)\sin\tfrac{1}{2}\epsilon\sin\omega t \tag{76}$$
for small angles ψ.

This last result may be written as
$$\psi = \psi_0 \sin\omega t, \tag{77}$$
so that
$$r = \dot{\psi} = \psi_0\omega\cos\omega t, \quad \dot{r} = \ddot{\psi} = -\psi_0\omega^2\sin\omega t. \tag{78}$$

The forces applied at P and Q during the sinusoidal yawing motion may be written in the form (65) again and we may take ψ as the reference for the measurement of phase.

The equations governing yawing motion are those for planar motion with terms $Y(t)$ and $N(t)$ introduced to denote the applied force and moment respectively. That is to say, the motion of the sinusoidally yawing model is governed by

$$\tilde{Y}_{\dot{r}}\psi_0\omega^2\sin\omega t+(m\overline{U}-\tilde{Y}_r)\psi_0\omega\cos\omega t = F_{\text{in}}\sin\omega t+F_{\text{out}}\cos\omega t, \tag{79}$$
$$-(I_z-\tilde{N}_{\dot{r}})\psi_0\omega^2\sin\omega t-\tilde{N}_r\psi_0\omega\cos\omega t = G_{\text{in}}\sin\omega t+G_{\text{out}}\cos\omega t. \tag{80}$$

By the same method as before we now find that

$$\tilde{Y}_{\dot{r}} = F_{\text{in}}/\psi_0\omega^2, \quad \tilde{Y}_r = m\overline{U}-F_{\text{out}}/\psi_0\omega, \tag{81}$$

from the force equation (79) and

$$\tilde{N}_{\dot{r}} = I_z+G_{\text{in}}/\psi_0\omega^2, \quad \tilde{N}_r = -G_{\text{out}}/\psi_0\omega, \tag{82}$$

from the moment equation (80).

These results, too, can be illustrated with data found by van Leeuwen. Once again it turns out that F_{in}, F_{out}, G_{in} and G_{out} are all proportional to ψ_0, for small frequencies ω—provided the motion falls within the limits already mentioned. For his model he found the curves shown in figure 10(*a*)–(*d*), which are all for Fn = 0.2. Notice that all the curves of figures 9 and 10 are flat near $\omega = 0$.

As in the pure sway test, it is also possible to calculate the coefficients $\tilde{L}_r$ and $\tilde{L}_{\dot{r}}$, if the components T_{in} and T_{out} of the torque applied to the model by the supporting structure are measured with the roll gauge. If

$$\Delta L = \tilde{L}_r r+\tilde{L}_{\dot{r}}\dot{r}, \quad L(t) = T(t) = T_{\text{in}}\sin\omega t+T_{\text{out}}\cos\omega t, \quad p = 0 \neq r, \tag{83}$$

in the second of equations (56), then

$$\tilde{L}_r\psi_0\omega\cos\omega t-\tilde{L}_{\dot{r}}\psi_0\omega^2\sin\omega t+T_{\text{in}}\sin\omega t+T_{\text{out}}\cos\omega t = I_{xz}\psi_0\omega^2\sin\omega t$$

whence
$$\tilde{L}_r = -T_{\text{out}}/\psi_0\omega, \quad \tilde{L}_{\dot{r}} = -I_{xz}+T_{\text{in}}/\psi_0\omega^2. \tag{84}$$

Rudder coefficients

It is normal practice in naval architecture to employ only slow motion derivatives and to disregard all of them except Y_ζ and N_ζ. The planar motion mechanism—or, to be more precise, its force transducers—permits one to measure the oscillatory coefficients $\tilde{Y}_\zeta$, $\tilde{Y}_{\dot{\zeta}}$, $\tilde{N}_\zeta$, $\tilde{N}_{\dot{\zeta}}$. From these, the four slow motion derivatives may be found; it may indeed then turn out that the $\dot{\zeta}$ derivatives are not significant.

The technique to be adopted is comparable with that employed for hydroplane coefficients. That is to say the model is towed straight along the tank with y_1 and y_2 fixed and equal to zero and with a suitable actuator giving the rudder a sinusoidal deflexion of known amplitude and frequency. If the deflexion is

$$\zeta = \zeta_0 \sin \omega t, \tag{85}$$

so that

$$\dot{\zeta} = \zeta_0 \omega \cos \omega t, \tag{86}$$

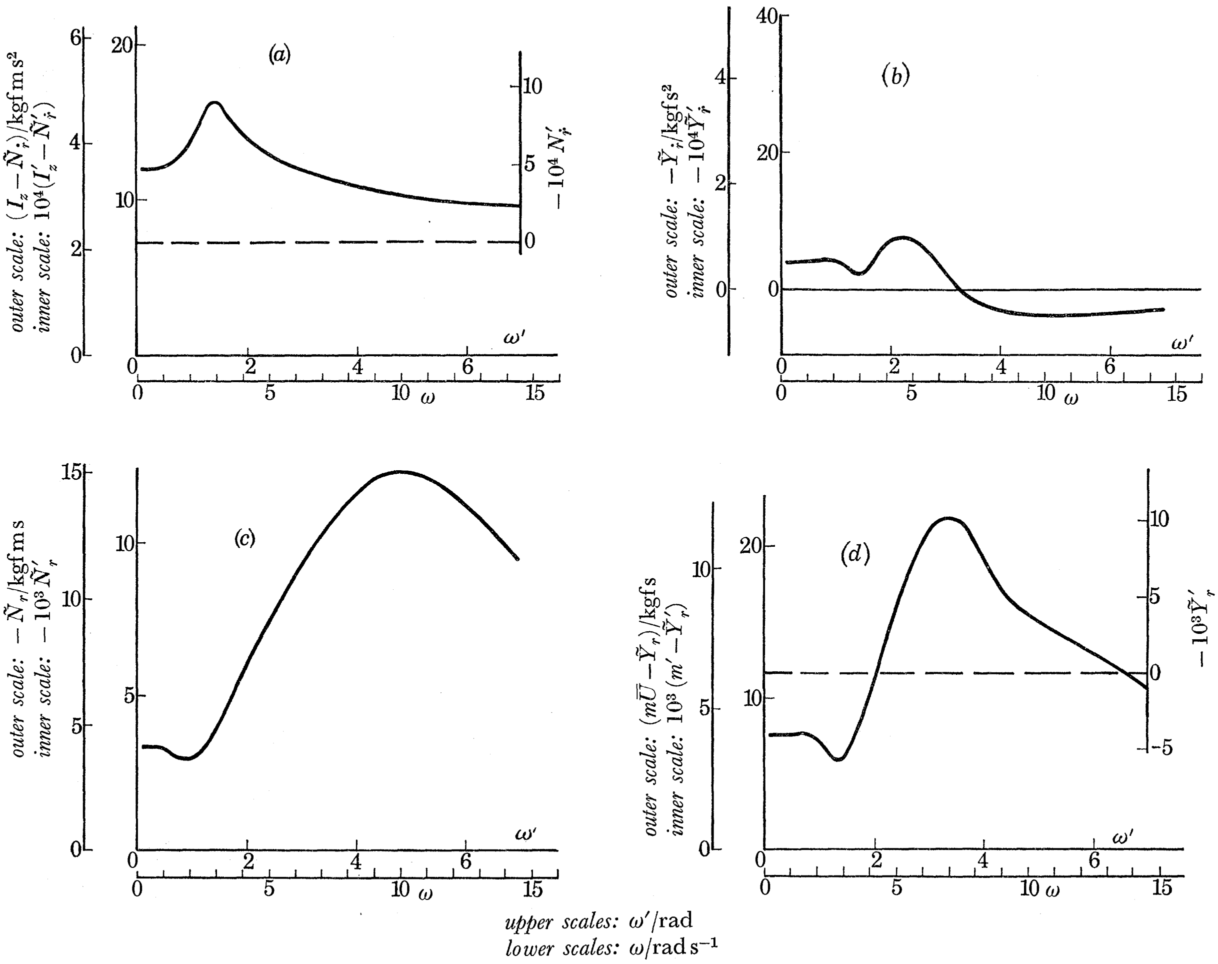

FIGURE 10

then the appropriate equations of motion are

$$\tilde{Y}_\zeta \zeta + \tilde{Y}_{\dot\zeta}\dot\zeta + [(F_1)_{\text{in}} + (F_2)_{\text{in}}]\sin \omega t + [(F_1)_{\text{out}} + (F_2)_{\text{out}}]\cos \omega t = 0, \tag{87}$$

$$\tilde{N}_\zeta \zeta + \tilde{N}_{\dot\zeta}\dot\zeta + l_0\{[(F_1)_{\text{in}}\sin \omega t + (F_1)_{\text{out}}\cos \omega t] - [(F_2)_{\text{in}}\sin \omega t + (F_2)_{\text{out}}\cos \omega t]\} = 0, \tag{88}$$

when allowance is made for the $\dot\zeta$ terms. Following the same line of argument as before, we now find that

$$\tilde{Y}_\zeta = -F_{\text{in}}/\zeta_0, \quad \tilde{Y}_{\dot\zeta} = -F_{\text{out}}/\zeta_0\omega, \tag{89}$$

from equation (87) and

$$\tilde{N}_\zeta = -G_{\text{in}}/\zeta_0, \quad \tilde{N}_{\dot\zeta} = -G_{\text{out}}/\zeta_0\omega \tag{90}$$

from equation (88).

When the rudder oscillates in this test,

$$\Delta L = \tilde{L}_\zeta \zeta + \tilde{L}_{\dot{\zeta}} \dot{\zeta}, \quad L(t) = T(t) = T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t, \quad p = 0 = r, \tag{91}$$

in the second of equations (56). That is

$$\tilde{L}_\zeta \zeta + \tilde{L}_{\dot{\zeta}} \dot{\zeta} + T_{\text{in}} \sin \omega t + T_{\text{out}} \cos \omega t = 0, \tag{92}$$

so that

$$\tilde{L}_\zeta = -T_{\text{in}}/\zeta_0, \quad \tilde{L}_{\dot{\zeta}} = -T_{\text{out}}/\zeta_0 \omega. \tag{93}$$

It is usual practice only to use the slow motion derivative L_ζ in the analysis of roll.

4. Conclusions

The planar motion mechanism (or PMM as it is often abbreviated) was pioneered in the U.S.A. and has been used successfully there for several years. With many successes to its credit, it is now regarded by many as being more of a production than a research tool. Although the underlying idea is quite straightforward and is strongly suggested by aeronautical practice, we in the U.K. have only recently installed our first such facility.

In these circumstances it would perhaps seem premature, if not presumptuous, for the present authors to write a paper at this stage on the PMM even though they have had the benefit of discussions with users of these facilities in the United States—discussions for which the authors are very grateful. There are, however, several arguments in favour of writing the present paper simply because we in the U.K. *are* only now entering the field. The reasons are:

(1) There is now a need for acquainting a wider circle of British naval architects with the uses, potentialities and shortcomings of the PMM.

(2) The authors suggest that the concept of oscillatory coefficients, as it has been developed here, has definite advantages over that of slow motion derivatives in the present context, both as an aid to thought and in logic.

(3) The oscillatory coefficients offer a simple approach to a matter that has already excited some discussion (see Brard 1964). The point is that, in theory, a wide range of transient problems of ship motion can be examined by means of Fourier integral techniques (see, for example, Mitchell (1964)) if the oscillatory coefficients can be found for an *infinite* range of frequency. Typically, the motion of a ship could perhaps be analysed when the ship moves from slack water into a tidal current. This aspect of ship dynamics is essentially something for the future. So far, little effort has been made to examine it systematically although it may have profound implications where hydrofoils and other high speed craft are concerned.

(4) There is evidence to suggest that the PMM is a sufficiently powerful tool eventually to supplant other techniques for measuring derivatives. Even so, it can be (and sometimes is) claimed that the case for the technique has not yet been fully made. It is perfectly legitimate, for instance, to inquire what published evidence there is that the values of Y_r and N_r found with a PMM agree with those measured with a rotating arm.† In fact basic questions of this sort are hardly touched on in the literature. Most workers in the field have been fully occupied in urgent design and development projects and have not had time to conduct the necessary fundamental studies. In short, it may be premature to describe the PMM as a production tool.

(5) Although the PMM was originally designed to measure slow motion derivatives for use in linear analysis of ship motions, it has been employed to measure coefficients specifying non-

† Some comparisons have been published for surface ship models by Chislett & Strøm-Tejsen (1965) and Motora & Fujino (1965).

linear force-motion relations. While it is undoubtedly desirable to widen the usefulness of the apparatus in this way, this side of things is very much in its infancy and offers great scope for research.

Appendix 1. Taylor series representation of disturbed motion

If a function $v(t)$ is sufficiently well behaved, then its value at any other instant $t-\tau$ can be expressed according to Taylor's theorem [see, for example, Hardy (1952)] in the form

$$v(t-\tau) = v(t) - \tau\dot{v}(t) + \frac{\tau^2}{2!}\ddot{v}(t) + \ldots + R_n, \tag{94}$$

where

$$R_n = (-1)^n \frac{\tau^n}{n!}\left|\frac{d^n v}{dt^n}\right|_{t=t-\tau\theta}$$

is Lagrange's form of the remainder, in which $0 < \theta < 1$.

In this paper we are concerned with motions for which

$$v = v_0 e^{\mu t} \sin(\omega t + \epsilon)$$

is a typical function. For a function of this form the modulus of R_n can be determined for all τ by the following inequality:

$$|R_n| < \frac{\tau^n}{n!}\left[\bar{\mu}^n + \binom{n}{1}\bar{\mu}^{n-1}\omega + \binom{n}{2}\bar{\mu}^{n-2}\omega^2 + \ldots + \binom{n}{r}\bar{\mu}^{n-r}\omega^r + \ldots + \omega^n\right] v_0 e^{(t-\theta\tau)} = \frac{\tau^n}{n!}(\bar{\mu}+\omega)^n v_0 e^{(t-\theta\tau)}$$

where $\bar{\mu} = |\mu|$. Therefore

$$\lim_{n\to\infty}|R_n| = v_0 e^{\mu(t-\theta\tau)} \lim_{n\to\infty}\frac{[\tau(\bar{\mu}+\omega)]^n}{n!} = 0.$$

In these circumstances the expansion (77) can be replaced by the Taylor series

$$v(t-\tau) = v(t) - \tau\dot{v}(t) + \frac{\tau^2}{2!}\ddot{v}(t) - \frac{\tau^3}{3!}\dddot{v}(t) + \ldots \tag{95}$$

and therefore equation (4) and similar expansions for the other motion parameters are justified.

Appendix 2. Positions of the centres of mass of the model and prototype

In this paper it has been assumed implicitly that the model used in PMM tests is a replica of the full-scale submarine or ship, not only as regards its hull form but also in the position of its centre of mass. In these circumstances the centre of buoyancy of the model (which coincides with the corresponding point in the vessel) lies on the C*z* axis. In practice, although the requirement for similar geometry is fairly easily met, there may be some difficulty in obtaining the correct mass distribution. It is, however, desirable that the model axes C*xyz* should be fixed in the model with their origin at the point corresponding to the centre of mass of the actual vessel. For in analysing the motions of full-size ships it is most convenient (and very common) to work in terms of body axes whose origin is located at the latter point. In this way the fluid coefficients are derived for a coordinate system in which the flow patterns for both model and ship are identical. Unfortunately the mass centre of the model may be offset from the origin C fixed in this manner, although it is still in the C*xz* plane of symmetry.

By contrast, it is occasionally desirable to fix the origin of the body axes at a point other than the mass centre (of both model and fullscale ship). In this way one can sometimes simplify the theoretical treatment of the hydrodynamics (see Abkowitz 1964), though the point will not be taken up here.

In fixing the body axes Cxyz in the model, therefore, the locations of the centres of mass and buoyancy of the model relative to C are very important. If the model and the actual ship have similar hull forms, their centres of buoyancy are at corresponding points. The model is unlikely, however, to possess a correctly scaled mass distribution by reason of the necessity to accommodate test equipment, etc. Corrections can of course be made to reduce this discrepancy, but complete correction is not to be expected, although the model centre of mass is usually in the Cxz plane of symmetry.

Suppose, therefore, that in a particular model the mass centre is located at a point A with co-ordinates $(\bar{x}, 0, \bar{z})$, whereas the origin C of the body axes is correctly situated to represent the mass centre of the full-size vessel. The centre of buoyancy is thus still on the Cz axis and the points P and Q in figures 3 and 5 to 7 are equally spaced about C. The linear equations of motion, representing forces and moments of forces about the point C (and not about the mass centre), may be shown to be as follows:

$$\left.\begin{aligned} m(\dot{u}+\dot{q}\bar{z})+mg\theta &= \Delta X+X(t),\\ m(\dot{w}-q\bar{U}-\dot{q}\bar{x}) &= \Delta Z+Z(t)+Z_{*},\\ I_y\dot{q}+m(\dot{u}\bar{z}-\dot{w}\bar{x}+q\bar{U}\bar{x})+mg\bar{z}\theta &= \Delta M+M(t)+M_{*};\end{aligned}\right\} \tag{96}$$

$$\left.\begin{aligned} m(\dot{v}+r\bar{U}+\dot{r}\bar{x}-\dot{p}\bar{z})-mg\phi &= \Delta Y+Y(t),\\ I_x\dot{p}-I_{xz}\dot{r}-m\bar{z}(\dot{v}+r\bar{U})+mg\bar{z}\phi &= \Delta L+L(t),\\ -I_{zx}\dot{p}+I_z\dot{r}+m\bar{x}(\dot{v}+r\bar{U})-mg\bar{x}\phi &= \Delta N+N(t);\end{aligned}\right\} \tag{97}$$

where the equations for symmetric and antisymmetric motion have been grouped separately. It should be noted that u, v, w refer now to the incremental components of velocity of the point C and not of the mass centre. The derivation of equations (96) and (97) is explained elsewhere (Bishop & Parkinson 1969).

The method of solution of these equations for the oscillatory coefficients corresponding to the various forms of imposed oscillatory motion is similar to that employed throughout the main body of the paper. We shall examine only one example here, but all the results are listed in tables 1 to 4.

Consider, for example, the pure heaving motion defined by equations (18) and (26). For this form of displacement excitation only the second and third of equations (96) are of interest and they reduce to

$$m\dot{w} = \Delta Z+Z(t)+Z_{*}, \tag{98}$$

$$-m\bar{x}\dot{w} = \Delta M+M(t)+M_{*}. \tag{99}$$

Equation (98), however, is independent of $\bar{x}$ and $\bar{z}$, so that the equations for the coefficients $\tilde{Z}_w$ and $\tilde{Z}_{\dot{w}}$ are not altered, if the mass centre of the model is off-set from the origin of the body axes.

If the usual substitutions are made in equation (99) this becomes

$$m\omega^2\bar{x}z_0\sin\omega t = \tilde{M}_w z_0\omega\cos\omega t-\tilde{M}_{\dot{w}}z_0\omega^2\sin\omega t+G_{*}+G_{\text{in}}\sin\omega t+G_{\text{out}}\cos\omega t+M_{*}, \tag{100}$$

so that the relevant oscillatory coefficients are given by

$$M_{*} = -G_{*}, \quad \tilde{M}_{w} = -G_{\text{out}}/z_0\,\omega, \quad \tilde{M}_{\dot{w}} = -m\bar{x}+G_{\text{in}}/z_0\,\omega^2. \tag{101}$$

The effect of displacing the mass centre in this case then is to modify the coefficient $\tilde{M}_{\dot{w}}$ by the inclusion of the $-m\bar{x}$ term, while the expressions for M_* and $\tilde{M}_w$ are unchanged. As noted above there is no point in explaining the detailed algebra for the other kinds of oscillatory motion, but all of the results are given in the appropriate columns of the accompanying tables.

Appendix 3. Tables of oscillatory coefficients

For convenience of reference the expressions for the various oscillatory coefficients are listed in tables 1–4. The tables include results (for the symmetric motion of a surface ship for instance) which are not derived in the main body of the paper, but which can be formulated by similar methods. All the oscillatory coefficients are for a model with an offset mass centre; if the centre of mass is not offset it is only necessary to set $\bar{x} = 0 = \bar{z}$. The dashes in the table indicate that the coefficients concerned are not defined.

Table 1. Oscillatory coefficients for symmetric motion

oscillatory coefficient	submerged vessel	vessel on surface
$\tilde{Z}_z$	—	$-m\omega^2-(F_{\text{in}}/z_0)$
$\tilde{Z}_w$	$-F_{\text{out}}/z_0\,\omega$	$-F_{\text{out}}/z_0\,\omega$
$\tilde{Z}_{\dot{w}}$	$m+(F_{\text{in}}/z_0\,\omega^2)$	—
$\tilde{Z}_\theta$	—	$m\bar{x}\omega^2-(F_{\text{in}}/\theta_0)$
$\tilde{Z}_q$	$-mU-(F_{\text{out}}/\theta_0\,\omega)$	$-mU-(F_{\text{out}}/\theta_0\,\omega)$
$\tilde{Z}_{\dot{q}}$	$-m\bar{x}+(F_{\text{in}}/\theta_0\,\omega^2)$	—
$\tilde{M}_z$	—	$m\bar{x}\omega^2-(G_{\text{in}}/z_0)$
$\tilde{M}_w$	$-G_{\text{out}}/z_0\,\omega$	$-G_{\text{out}}/z_0\,\omega$
$\tilde{M}_{\dot{w}}$	$-m\bar{x}+G_{\text{in}}/z_0\,\omega^2$	—
$\tilde{M}_\theta$	$mg\bar{z}-I_y\omega^2-(G_{\text{in}}/\theta_0)$	$mg\bar{z}-I_y\omega^2-(G_{\text{in}}/\theta_0)$
$\tilde{M}_q$	$mU\bar{x}-(G_{\text{out}}/\theta_0\,\omega)$	$mU\bar{x}-(G_{\text{out}}/\theta_0\,\omega)$

Table 2. Oscillatory coefficients for antisymmetric motion

Vessel submerged or on surface

oscillatory coefficient		oscillatory coefficient	
$\tilde{Y}_v$	$-F_{\text{out}}/y_0\,\omega$	$\tilde{N}_r$	$m\bar{x}U-(G_{\text{out}}/\psi_0\,\omega)$
$\tilde{Y}_{\dot{v}}$	$m+(F_{\text{in}}/y_0\,\omega^2)$	$\tilde{N}_{\dot{r}}$	$I_z+(G_{\text{in}}/\psi_0\,\omega^2)$
$\tilde{Y}_r$	$mU-(F_{\text{out}}/\psi_0\,\omega)$	$\tilde{L}_v$	$-T_{\text{out}}/y_0\,\omega$
$\tilde{Y}_{\dot{r}}$	$m\bar{x}+(F_{\text{in}}/\psi_0\,\omega^2)$	$\tilde{L}_{\dot{v}}$	$-m\bar{z}+(T_{\text{in}}/y_0\,\omega^2)$
$\tilde{N}_v$	$-G_{\text{out}}/y_0\,\omega$	$\tilde{L}_r$	$-m\bar{z}U-(T_{\text{out}}/\psi_0\,\omega)$
$\tilde{N}_{\dot{v}}$	$m\bar{x}+(G_{\text{in}}/y_0\,\omega^2)$	$\tilde{L}_{\dot{r}}$	$-I_{xz}+(T_{\text{in}}/\psi_0\,\omega^2)$

TABLE 3. OSCILLATORY COEFFICIENTS FOR ROLLING MOTION

oscillatory coefficient	submerged vessel	vessel on surface
$\tilde{Y}_\phi$	$m\bar{z}\omega^2 - mg - (P_{\text{in}}/\phi_0)$	$m\bar{z}\omega^2 - mg - (P_{\text{in}}/\phi_0)$
$\tilde{Y}_p$	$-P_{\text{out}}/\phi_0\omega$	$-P_{\text{out}}/\phi_0\omega$
$\tilde{N}_\phi$	—	$I_{zx}\omega^2 - mg\bar{x} - (H_{\text{in}}/\phi_0)$
$\tilde{N}_p$	$-H_{\text{out}}/\phi_0\omega$	$-H_{\text{out}}/\phi_0\omega$
$\tilde{N}_{\dot{p}}$	$-I_{zx} + (mg\bar{x}/\omega^2) + (H_{\text{in}}/\phi_0\omega^2)$	—
$\tilde{L}_\phi$	$mg\bar{z} - I_x\omega^2 - (T_{\text{in}}/\phi_0)$	$mg\bar{z} - I_x\omega^2 - (T_{\text{in}}/\phi_0)$
$\tilde{L}_p$	$-T_{\text{out}}/\phi_0\omega$	$-T_{\text{out}}/\phi_0\omega$

TABLE 4. OSCILLATORY COEFFICIENTS FOR CONTROL SURFACES

For any position of the model mass centre

oscillatory coefficient	submerged vessel	vessel on surface
$\tilde{Y}_\zeta$	$-F_{\text{in}}/\zeta_0$	$-F_{\text{in}}/\zeta_0$
$\tilde{Y}_{\dot{\zeta}}$	$-F_{\text{out}}/\zeta_0\omega$	$-F_{\text{out}}/\zeta_0\omega$
$\tilde{Z}_\eta$	$-F_{\text{in}}/\eta_0$	—
$\tilde{Z}_{\dot{\eta}}$	$-F_{\text{out}}/\eta_0\omega$	—
$\tilde{L}_\zeta$	$-T_{\text{in}}/\zeta_0$	$-T_{\text{in}}/\zeta_0$
$\tilde{L}_{\dot{\zeta}}$	$-T_{\text{out}}/\zeta_0\omega$	$-T_{\text{out}}/\zeta_0\omega$
$\tilde{M}_\eta$	$-G_{\text{in}}/\eta_0$	—
$\tilde{M}_{\dot{\eta}}$	$-G_{\text{out}}/\eta_0\omega$	—
$\tilde{N}_\zeta$	$-G_{\text{in}}/\zeta_0$	$-G_{\text{in}}/\zeta_0$
$\tilde{N}_{\dot{\zeta}}$	$-G_{\text{out}}/\zeta_0\omega$	$-G_{\text{out}}/\zeta_0\omega$

It will be understood that the expressions listed in table 1 for a surface ship are formulated in terms of the body axes C*xyz*. In particular the oscillatory coefficients in pitch should be regarded as coefficients relating to the force and moment components in phase and in quadrature with $\theta_0 \sin \omega t$. Values for the corresponding slow motion derivatives must be deduced with caution from experimental results as the fluid loadings depend on the depth of immersion of the point C as well as on the pitch parameters $\theta, q, \dot{q}, \ldots$. In practice one would normally adopt non-rotating moving axes for an analysis of the symmetric motions of a surface ship, but the necessary additional theory is beyond the scope of this paper and is better treated separately.

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